Recent extensions of Normaliz

Winfried Bruns

FB Mathematik/Informatik
Universität Osnabrück

wbruns@uos.de

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Basic objects of Normaliz

The basic objects that constitute the input of Normaliz are:

- C, a finitely generated rational cone in $\mathbb{R}^d$
- L, a sublattice of $\mathbb{Z}^d$

Normaliz computes the monoid

$$M = C \cap L$$

It is finitely generated by Gordan's lemma.

Normaliz has applications in commutative algebra, toric geometry, combinatorics, integer programming, invariant theory, elimination theory, mathematical logic, algebraic topology, theoretical physics

Recent extension: $P \cap L$ where P is a rational polyhedron

Platforms and systems

Normaliz (present public version 2.10.1) is written in C++ (using Boost and GMP/MPIR), parallelized with OpenMP, and runs on

- Apple
- Linux
- MS Windows

Access to Normaliz from (partly via libnormaliz)

- Singular (library by WB and Christof Söger)
- Macaulay 2 (package by Gesa Kämpf)
- CoCoA (via library interface; John Abbott, Anna Bigatti, Christof Söger)
- Sage (optional package by Andrey Novoseltsev)
- polymake (polymake team)
- Regina (for 3-manifolds, by Benjamin Burton)

GUI interface jNormaliz (by V. Almendra and B. Ichim)

Input to Normaliz

Cones C and lattices L can be specified by

- generators $x_1, \ldots, x_n \in \mathbb{Z}^d$,
- constraints: homogeneous systems of diophantine linear inequalities, equations and congruences,
- relations: binomial equations.

Input for 3x3 magic squares with even corners:

```
7 9
1 1 1 -1 -1 -1  0  0  0
1 1 1  0  0  0 -1 -1 -1
0 1 1 -1  0  0 -1  0  0
1 0 1  0 -1  0  0 -1  0
1 1 0  0  0 -1  0  0 -1
0 1 1  0 -1  0  0  0 -1
1 1 0  0 -1  0 -1  0  0
equations
```

```
4 10
1 0 0 0 0 0 0 0 0 2
0 0 1 0 0 0 0 0 0 2
0 0 0 0 0 0 1 0 0 2
0 0 0 0 0 0 0 0 1 2
congruences
1 9
1 1 1 0 0 0 0 0 0
grading
```

The tasks of Normaliz: Hilbert bases

Normaliz computes (together with other data)

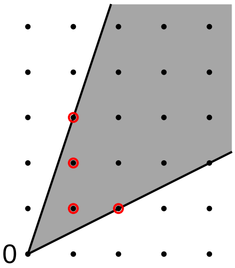
$$\mathrm{Hilb}(M), \qquad M = C \cap L$$

where $\mathrm{Hilb}(M)$ is the unique minimal system of generators of M. (Standard asumption: C is pointed.)

Important variant: lattice points in polytopes.

Algoritms (choice left to the user):

- the primal original Normaliz algorithm (based on pyramid decompositions and triangulations)
- the dual algoritm, a variant of an algorithm due to Pottier (of type pair completion)

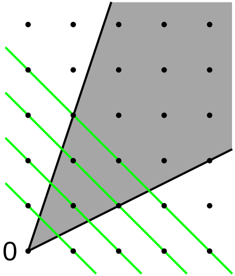


The tasks of Normaliz: Hilbert series

A grading on M is a surjective $\mathbb{Z}$-linear form $\deg : \mathrm{gp}(M) \to \mathbb{Z}$ such that $\deg(x) > 0$ for $x \in M$, $x \neq 0$

The Hilbert (or Ehrhart) function is given by

$$H(M,k) = \#\{x \in M : \deg x = k\}$$



and the Hilbert (Ehrhart) series is

$$H_M(t) = \sum_{k=0}^{\infty} H(M,k)t^k.$$

Algorithm: Stanley decomposition of monoid

Theorem (Hilbert-Serre, Ehrhart)

- *$H_M(t)$ is a rational function*
- *$H(M,k)$ is a quasi-polynomial for $k \geq 0$*

Extension I: Hilbert series of semiopen cones

A semiopen cone is given by

$$C' = C \setminus \mathscr{F}$$

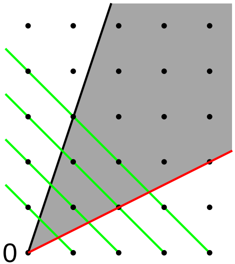
where C is a cone and $\mathscr{F}$ is a union of faces (not necessarily facets) of C. Set $M' = C' \cap L$

The task is to compute the Hilbert function

$$H(M', k) = \#\{x \in M' : \deg x = k\}$$

encoded by the Hilbert series

$$H_{M'}(t) = \sum_{k=0}^{\infty} H(M', k) t^k.$$



Typical application: mixed systems of homogeneous inequalities and strict inequalities

Algorithmic variant: approximation of rational polytopes

Huge determinants of simplicial cones can be a severe problem in the Normaliz primal algorithm even when computing lattice points in rational polytopes P. Vertices of an “innocent” polytope:

```
2 6 3 4 3 6 2 6 3 4
2 -3 3 -2 3 -3 2 -3 3 -2
4 -6 -3 14 -3 -6 7 -6 -3 8
4 3 -3 -7 -3 3 7 3 -3 -4
14 24 -6 -14 -33 -30 -7 6 39 28
14 6 -33 -14 39 24 -7 -30 -6 28
14 -30 39 -14 -6 6 -7 24 -33 28
14 15 12 7 3 -3 -7 -12 -15 -14
14 -3 -15 7 12 -12 -7 15 3 -14
14 -12 3 7 -15 15 -7 -3 12 -14
```

determinant 416,728,074,151,872

44 lattice points

denominator: column 1

A way out: approximate P by a lattice overpolytope Q , compute its lattice points, and select those in P.

Extension II: Polyhedra

Let $P \subset \mathbb{R}^d$ be a rational polydron (with vertices). There are two ways to describe it:

- P is the intersection of finitely many rational affine halfspaces
- $P = Q + C$ where Q is a rational polytope and C is a rational cone.

Since Q is the convex hull of finitely many points and C is finitely generated, P has a description by finitely many generators. We call C the recession cone.

One of the tasks of Normaliz is the conversion between both descriptions (like in the case of cones).

An affine lattice L is a Minkowski sum $x + L_0$ where $x \in \mathbb{Z}^d$ and $L_0 \subset \mathbb{Z}^d$ is a sublattice.

Lattice points in polyhedra

Let $P = Q + C$ be a rational polyhedron, L an affine lattice and $N = P \cap L$:

Proposition

There exist $x_1, \dots, x_m \in N$ such that

$$N = (x_1 + (L_0 \cap C)) \cup \dots \cup (x_m + (L_0 \cap C)),$$

and $x_1, \dots, x_m$ are uniquely determined if the union is irredundant.

We call $x_1, \dots, x_m$ the minimal system of generators of N.

Polyhedra C and affine lattices L can be specified by

- generators, integer vectors generating C and rational vertices of Q
- constraints: inhomogeneous system of diophantine linear inequalities, equations and congruences

Polyhedra: tasks of Normaliz

Normaliz computes

- the Hilbert basis of C
- the minimal system of generators of N

Furthermore, when a grading is given:

- the Hilbert series $H_N(t) = \sum_{k\in\mathbb{Z}} H(N,k)t^k$

Input for the interior of a cone in dimension 24:

```
1 24
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
strict_signs
3 24
1 1 1 1 1 1 -1 -1 -1 -1 -1 -1 1 1 -1 -1 1 -1 1 1 -1 -1 1 -1
1 1 1 1 1 1 1 1 -1 -1 1 -1 -1 -1 -1 -1 -1 -1 1 1 1 -1 -1 -1
1 1 1 1 1 1 1 1 1 -1 -1 -1 1 1 1 -1 -1 -1 -1 -1 -1 -1 -1 -1
strict_inequalities
```

Extension III: NmzIntegrate

NmzIntegrate is an executable of its own, but uses data produced by Normaliz. It is based on CoCoALib (A. Bigatti, J. Abbott).

NmzIntegrate computes

- integrals
$$\int_P f(x)\,dx$$
of polynomials $f \in \mathbb{Q}[x_1,\dots,x_d]$ over rational polytopes $P \subset \mathbb{R}^d$
- generalized (weighted) Ehrhart series
$$H_{M,f}(t) = \sum_{x\in M} f(x)t^{\deg x}$$
where $M = C \cap L$ as above (also in the semiopen case)
$H_{M,f}(t)$ is again a rational function.

Example: an integral

A recent paper of Jeffries, Montaño and Varbaro asks for the computation of

$$\int_{\substack{[0,1]^m \\ \sum x = t}} (x_1 \cdots x_m)^{n-m} \prod_{1 \le i < j \le m} (x_j - x_i)^2 \mathrm{d}x = \frac{27773}{29515186701000}$$

Result for $m = 4$, $n = 6$, $t = 2$ with input:

```
8                   0 0 0 1 0              1
5                   -1 0 0 0 1             5
1 0 0 0 0           0 -1 0 0 1             -1 -1 -1 -1 2
0 1 0 0 0           0 0 -1 0 1             equations
0 0 1 0 0           0 0 0 -1 1
                    inequalities
```

```
(x[1]*x[2]*x[3]*x[4])^2*(x[1]-x[2])^2*(x[1]-x[3])^2*
(x[1]-x[4])^2*(x[2]-x[3])^2*(x[2]-x[4])^2*(x[3]-x[4])^2
```

Example: A generalized Ehrhart series

Typical application of generalized Ehrhart series: Instead of counting lattice points in a high dimensional monoid one counts them in a low dimensional projection, but each with its number of preimages.

Can be applied to combinatorial voting theory, for example to the Condorcet paradox for 4 candidates (A. Schürmann). Instead of computing in Dimension 24 one can work with the projection:

```
1
8
1 1 1 1 1 1 1 1
signs
1
8
1 1 1 1 1 1 1 1
grading
```

```
3
8
1 -1 1 1 1 -1 -1 -1
1 1 -1 1 -1 1 -1 -1
1 1 1 -1 -1 -1 1 -1
excluded_faces
```

$$f(x) = \binom{x_1+5}{5}(x_2+1)(x_3+1)(x_4+1)(x_5+1)(x_6+1)(x_7+1)\binom{x_8+5}{5}$$

NmzIntegrate: methods

Integrals are computed by classical approach:

- triangulation of polytope P
- transformation of each simplex to the unit simplex
- explicit formula for the integration of a monomial over the unit simplex:

$$\int_{\Delta} y_1^{m_1} \cdots y_d^{m_d} \, d\mu = \frac{m_1! \cdots m_d!}{(m_1 + \cdots + m_d + d - 1)!}.$$

Ehrhart series use a completely analogous approach:

- Stanley decomposition of the monoid
- transformation to $\mathbb{Z}_+^d$ (the monoid over he unit $(d-1)$-simplex)
- Fubini type reduction to $d = 1$
- powers of the geometric series in $d = 1$

Outlook

- algorithmic improvements
- improvement of input and output
- massive parallelization
- reusability of partial results
- intelligent choice of integer type and algorithm
- exploitation of symmetries
- Graber bases
- multigraded Hilbert series

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